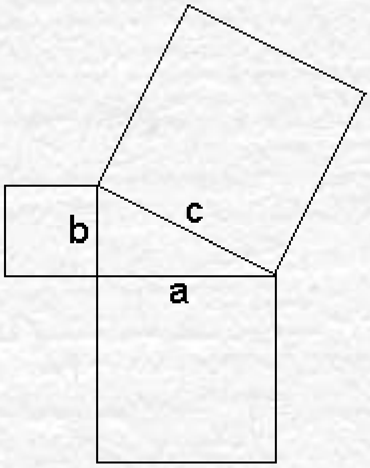
Πυθαγόρειο Θεώρημα



Το θεώρημα του Πυθαγόρα υποστηρίζει ότι το άθροισμα των τετραγώνων των δύο καθέτων πλευρών ενός ορθογωνίου τριγώνου είναι ίσο με το τετράγωνο της υποτείνουσας.

Με αλγεβρικούς όρους $\mathbf{a^2 + b^2 = c^2}$ όπου c η υποτείνουσα και a , b οι κάθετες πλευρές του ορθογωνίου τριγώνου..

Το θεώρημα είναι θεμελιώδους σπουδαιότητας στη Ευκλείδεια γεωμετρία όπου χρησιμεύει ως μια βάση για τον καθορισμό της απόστασης μεταξύ δύο σημείων. Είναι τόσο βασικό και γνωστό ώστε, πιστεύω, καθένας που πήρε τις γνώσεις γεωμετρίας στο γυμνάσιο δεν θα μπορούσε να μην το θυμηθεί ενώ άλλες έννοιες μαθηματικών έχουν ξεχαστεί.

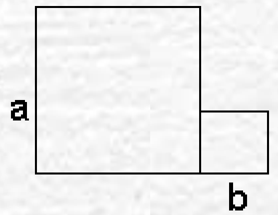
Παρακάτω παρουσιάζονται μερικές αποδείξεις του Πυθαγορείου θεωρήματος.

Παρατήρηση

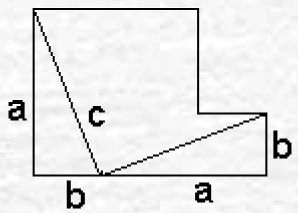
1. Μία αναφορά του θεωρήματος ανακαλύφθηκε σε μία Βαβυλωνιακή πινακίδα του 1900-1600 π.Χ. Όμως **ο Πυθαγόρας (560-480 π.Χ.) είτε κάποιος άλλος από τη σχολή του ήταν ο πρώτος που ανακάλυψε την απόδειξή του** δεν μπορούμε όμως να απαιτήσουμε οποιοδήποτε βαθμό αυστηρότητας. Ο Ευκλείδης (300 πΧ.) στα Στοιχεία του θέτει για πρώτη φορά τις βάσεις για την αυστηρή οργάνωση της γεωμετρίας. **Το θεώρημα είναι αντιστρέψιμο** που σημαίνει ότι ένα τρίγωνο οι του οποίου πλευρές ικανοποιούν την σχέση $a^2+b^2=c^2$ είναι ορθογώνιο. **Ο Ευκλείδης ήταν ο πρώτος που αναφέρει και αποδεικνύει αυτό το γεγονός.**
2. W.Dunham [*Mathematical Universe*] Αναφέρει το βιβλίο *The Pythagorean Proposition* by an early 20th century professor Elisha Scott Loomis. Το βιβλίο είναι μια συλλογή **367 αποδείξεων** του πυθαγορίου θεωρήματος και έχει αναδημοσιευτεί από NCTM το 1968.
3. Το πυθαγόρειο θεώρημα γενικεύεται στα διαστήματα των υψηλότερων διαστάσεων. Μερικές από τις γενικεύσεις δεν είναι προφανείς.
4. Το θεώρημα η του οποίου διατύπωση οδηγεί στην έννοια της Ευκλείδειας απόστασης και Ευκλείδειων χώρων και χώρων Hilbert, διαδραματίζει έναν σημαντικό ρόλο στα μαθηματικά συνολικά.

5. Οταν και οι τρεις πλευρές ενός ορθογωνίου τριγώνου είναι ακέραιοι αριθμοί, τα μήκη τους σχηματίζουν μία Πυθαγόρεια τριάδα (ή τους πυθαγόρειους αριθμούς). Υπάρχει ένας γενικός τύπος για τη λήψη όλων αυτών των αριθμών.

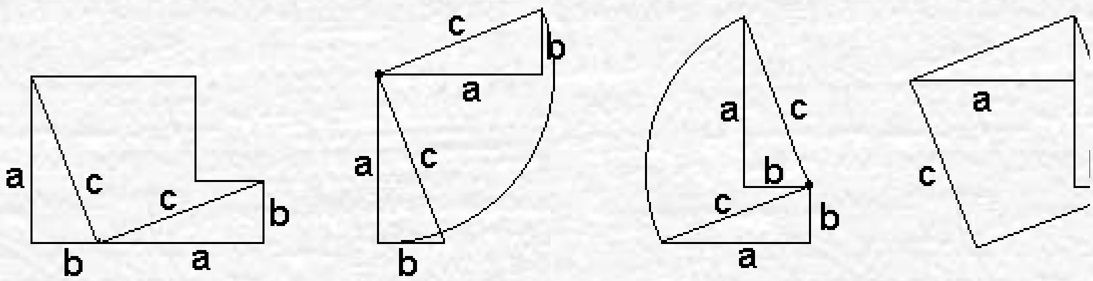
Απόδειξη #1



Αρχίζουμε με δύο τετράγωνα με τις πλευρές a και b, αντίστοιχα, που τοποθετούνται δίπλα-δίπλα. συνολικό εμβαδόν των δύο τετραγώνων είναι $\mathbf{a^2+b^2}$.



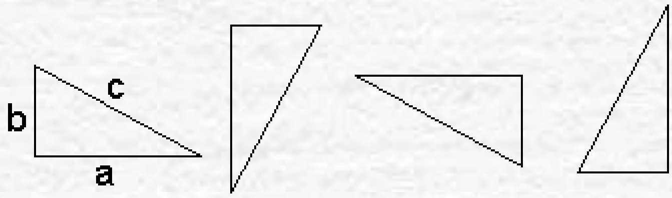
Η κατασκευή δεν άρχισε με ένα τρίγωνο αλλά τώρα σχεδιάζουμε δύο ορθογώνια τρίγωνα, και με τις πλευρές a και b και με υποτείνουσα c. Αφαιρούμε το κοινό μέρος των δύο τετραγώνων. Σε αυτό το σημείο επομένως έχουμε δύο τρίγωνα.



Σαν τελευταίο βήμα, περιστρέφουμε τα τρίγωνα 90^o, κάθε ένα γύρω από της κορυφές. Το δεξιό περιστρέφεται δεξιόστροφα ενώ το αριστερό τρίγωνο περιστρέφεται αντίθετα προς τη φορά των δεικτων του ρολογιού. Προφανώς η προκύπτουσα μορφή είναι ένα τετράγωνο με πλευρές c και εμβαδόν $\mathbf{c^2}$.

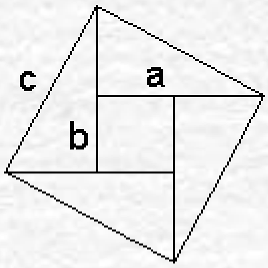
Απόδειξη #2

Τώρα αρχίζουμε με τέσσερα αντίγραφα του ίδιου τριγώνου. Τρία από αυτά έχουν περιστραφεί κατά 90^o, 180^o, και 270^o, αντίστοιχα. Κάθε ένα έχει εμβαδόν ab/2. Βάλτε τα μαζί χωρίς πρόσθετες περιστροφές έτσι ώστε διαμορφώνουν ένα τετράγωνο με το πλευρά c.



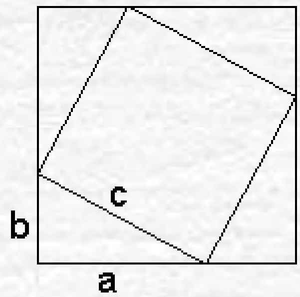
Το τετράγωνο έχει μια τετραγωνική τρύπα με την πλευρά (**a-b**). και εμβαδόν **$(a-b)^2$** και **2ab** είναι το εμβαδόν των τεσσάρων τριγώνων **(4ab/2)**,
έτσι παίρνουμε :

$$\mathbf{c^2 = (a-b)^2+2ab = a^2-2ab+b^2+2ab = a^2+b^2}$$



Απόδειξη #3

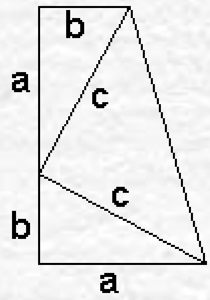
Η τρίτη προσέγγιση αρχίζει με τα ίδια τέσσερα τρίγωνα, εκτός από το ότι, αυτή τη φορά, συνδυάζονται ώστε να διαμορφώσουν ένα τετράγωνο με την πλευρά (a+b) και μια τρύπα με πλευρά c. Μπορούμε να υπολογίσουμε τον εμβαδόν του μεγάλου τετραγώνου με δύο τρόπους. Κατά συνέπεια



$$(\mathbf{a + b})^2 = 4\cdot\mathbf{ab}/2 + \mathbf{c}^2$$

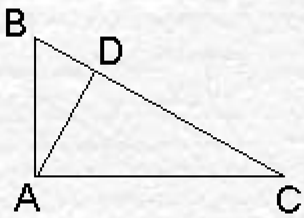
απλοποιώντας παίρνουμε την ζητούμενη σχέση.

Απόδειξη#4



Αυτή η απόδειξη, που ανακαλύφθηκε από τον Πρόεδρο J.A. Garfield το 1876 [Pappas], είναι μια παραλλαγή στην προηγούμενη. Αλλά αυτή τη φορά δεν σύρουμε κανένα τετράγωνο καθόλου. Το κλειδί είναι τώρα ο τύπος για το εμβαδόν τραπεζίου (ημιάθροισμα των βάσεων επί το ύψος) (**a+b**)/2·(**a+b**). Εξετάζοντας την εικόνα ένας άλλος τρόπος, που το εμβαδόν μπορεί επίσης να υπολογιστεί ως άθροισμα των εβαδών των τριών τριγώνων **ab**/2 + **ab**/2 + **c**·**c**/2. Όπως πριν, οι απλοποιήσεις δίνουν $\mathbf{a^2+b^2=c^2}$.

Απόδειξη #5



Αρχίζουμε με το αρχικό τρίγωνο, ABC, και χρειάζεται μόνο βοηθητικά το ύψος. Τα τρίγωνα ABC, BDA και ADC είναι όμοια έτσι παίρνουμε δύο αναλογίες:

AB/BC = BD/AB and AC/BC = DC/AC.

Πολλαπλασιάζοντας "χιαστί" παίρνουμε

AB·AB = BD·BC and AC·AC = DC·BC

προσθέτοντας κατά μέλη

AB·AB + AC·AC = BD·BC + DC·BC = (BD+DC)·BC = BC·BC.

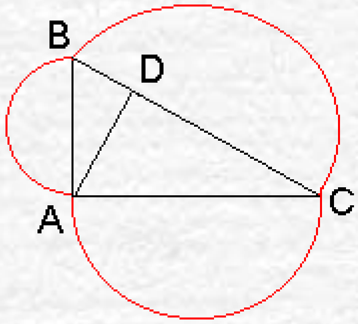
Proof #6

The next proof is taken verbatim from Euclid VI.31 in translation by Sir Thomas L. Heath. The great G. Polya analyzes it in his *Induction* *and Analogy in Mathematics* (II.5) which is a recommended reading to students and teachers of Mathematics.

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle BAC right; I say that the figure on BC is equal to the similar and similarly described figures on BA, AC.

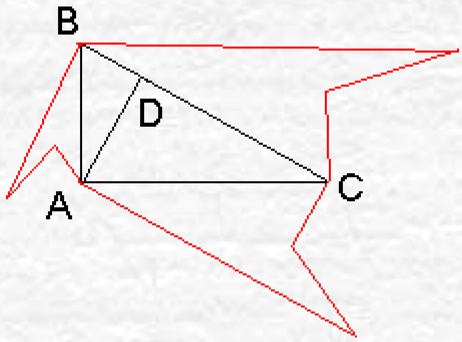
Let AD be drawn perpendicular. Then since, in the right-angled triangle ABC, AD has been drawn from the right angle at A perpendicular to the base BC, the triangles ABD, ADC adjoining the perpendicular are similar both to the whole ABC and to one another [VI.8].



And, since ABC is similar to ABD, therefore, as CB is to BA so is AB to BD [VI.Def.1]

And, since three straight lines are proportional, as the first is to the third, so is the figure on the first to the similar and similarly described figure on the second [VI.19]. Therefore, as CB is to BD, so is the figure on CB to the similar and similarly described figure on BA.

For the same reason also, as BC is to CD, so is the figure on BC to that on CA; so that, in addition, as BC is to BD, DC, so is the figure on BC to the similar and similarly described figures on BA, AC.



But BC is equal to BD, DC; therefore the figure on BC is also equal to the similar and similarly described figures on BA, AC.

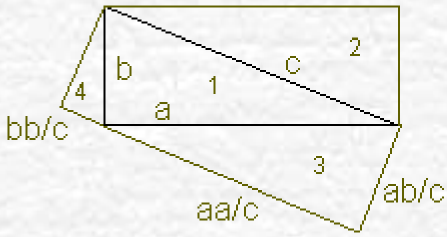
Therefore etc. Q.E.D.

Confession

I got a real appreciation of this proof only after reading the book by Polya I mentioned above. I hope that a Java applet will help you get to the bottom of this remarkable proof. Note that the statement actually proven is much more general than the theorem as it's generally known.

Proof #7

Playing with the applet that demonstrates the Euclid's proof (#7), I have discovered another one which, although ugly, serves the purpose nonetheless.



Thus starting with the triangle 1 we add three more in the way suggested in proof #7: similar and similarly described triangles 2, 3, and 4. Deriving a couple of ratios as was done in proof #6 we arrive at the side lengths as depicted on the diagram. Now, it's possible to look at the final shape in two ways:

- as a union of the rectangle (1+3+4) and the triangle 2, or
- as a union of the rectangle (1+2) and two triangles 3 and 4.

Equating areas leads to

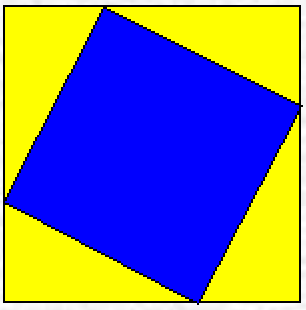
$$ab/c \cdot (a^2+b^2)/c + ab/2 = ab + (ab/c \cdot a^2/c + ab/c \cdot b^2/c)/2$$

Simplifying we get

$$ab/c \cdot (a^2+b^2)/c/2 = ab/2, \text{ or } (a^2+b^2)/c^2 = 1$$

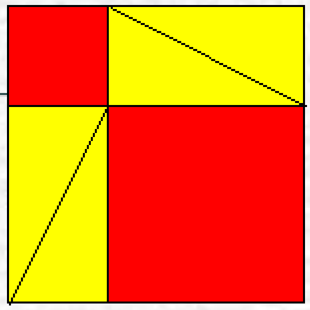
Remark

On a second look at the diagram, there is a simpler proof. Viz., look at the rectangle (1+3+4). Its long side is, on one hand, plain c while, on the other, it's $a^2/c+b^2/c$ and we again have the same identity.



Proof #8

Another proof stems from a

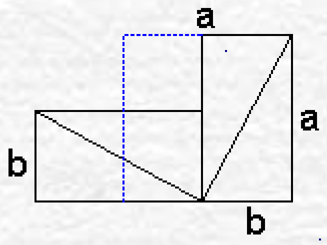


rearrangement of rigid pieces, much like proof #2. It makes the algebraic part of proof #4 completely redundant. There is nothing much one can add to the two pictures.

(My sincere thanks go to Monty Phister for the kind permission to use the graphics.)

There is an interactive simulation to toy with.

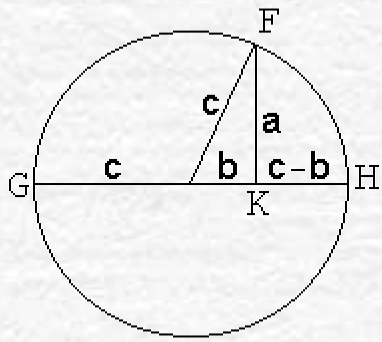
Proof #9



This and the next 3 proofs came from [*PWW*].

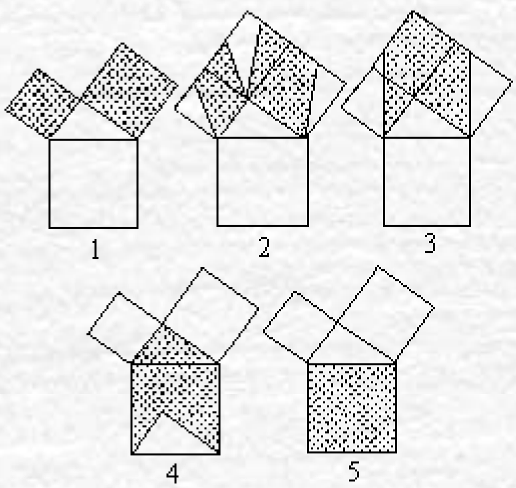
The triangles in Proof #3 may be rearranged in yet another way that makes the Pythagorean identity obvious.

Proof #10



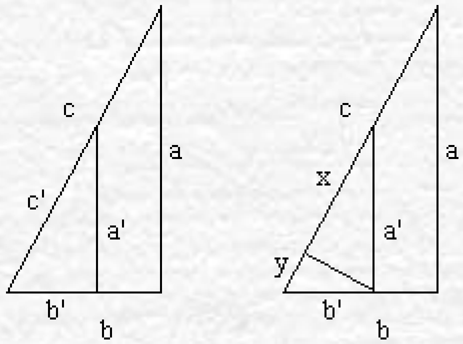
Draw a circle with radius c and a right triangle with sides a and b as shown. In this situation, one may apply any of a few well known facts. For example, in the diagram three points F, G, H located on the circle form another right triangle with the altitude FK of length a. Its hypotenuse GH is split in the ratio (c+b)/(c-b). So, as in Proof #6, we get $a^2 = (c+b)(c-b) = c^2 - b^2$.

Proof #11



This proof is a variation on #1, one of the original Euclid's proofs. In parts 1,2, and 3, the two small squares are sheared towards each other such that the total shaded area remains unchanged (and equal to a^2+b^2.) In part 3, the length of the vertical portion of the shaded area's border is exactly c because the two leftover triangles are copies of the original one. This means one may slide down the shaded area as in part 4. From here the Pythagorean Theorem follows easily.

Proof #12

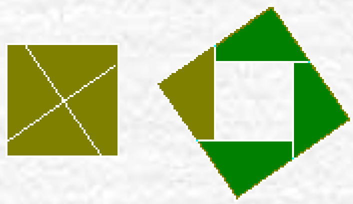


In the diagram there is several similar triangles (abc, a'b'c', b'x, and a'c'.) We successively have

$$y/b = b'/c,\ x/a = a'/c,\ cy + cx = aa' + bb'.$$

And, finally, cc' = aa' + bb'. This is very much like Proof #6 but the result is more general.

Proof #13



This proof by H.E.Dudeney (1917) starts by cutting the square on the larger side into four parts that are then combined with the smaller one to form the square built on the hypotenuse.

Greg Frederickson from Purdue University, the author of a truly illuminating book, *Dissections: Plane & Fancy* (Cambridge University Press, 1997), pointed out the historical inaccuracy:

> You attributed proof #14 to H.E. Dudeney (1917), but it was actually published earlier (1873) by Henry Perigal, a London stockbroker. A different dissection proof appeared much earlier, given by the Arabian mathematician/astronomer Thabit in the tenth century. I have included details about these and other dissections proofs (including proofs of the Law of Cosines) in my recent book "Dissections: Plane & Fancy", Cambridge University Press, 1997. You might enjoy the web page for the book:

http://www.cs.purdue.edu/homes/gnf/book.html

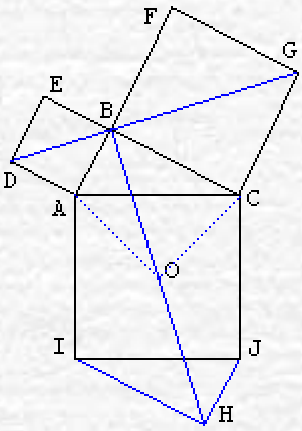
Sincerely,
Greg Frederickson

Bill Casselman from the University of British Columbia seconds Greg's information. Mine came from *Proofs Without Words* by R.B.Nelsen (MAA, 1993).

Proof #14

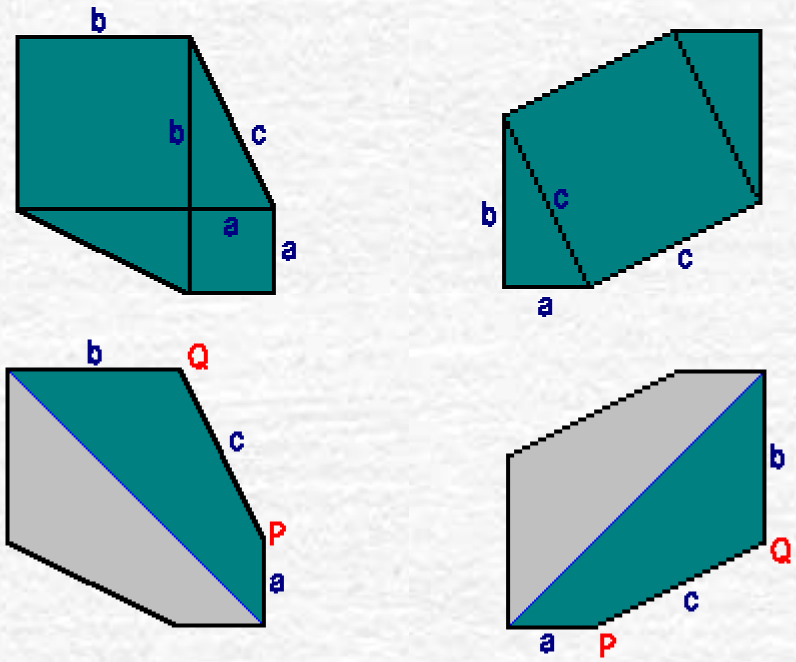
This remarkable proof by K.O.Friedrichs is a generalization of the previous one by Dudeney. It's indeed general. It's general in the sense that an infinite variety of specific geometric proofs may be derived from it. (Roger Nelsen ascribes [PWWII, p 3] this proof to Annairizi of Arabia (ca. 900 A.D.))

Proof #15



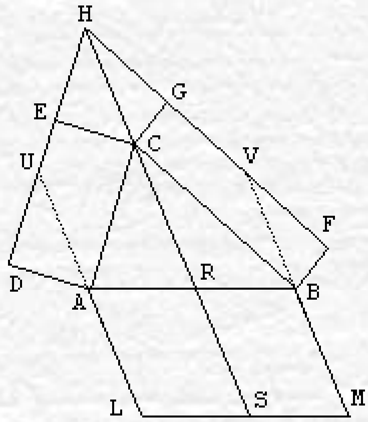
This proof is ascribed to Leonardo da Vinci (1452-1519) [Eves]. Quadrilaterals ABHI, JHBC, ADGC, and EDGF are all equal. (This follows from the observation that the angle ABH is 45°. This is so because ABC is right-angled, thus center O of the square ACJI lies on the circle circumscribing triangle ABC. Obviously, angle ABO is 45°.) Now, area(ABHI)+area(JHBC)=area(ADGC)+area(EDGF). Each sum contains two areas of triangles equal to ABC (IJH or BEF) removing which one obtains the Pythagorean Theorem.

David King modifies the argument somewhat



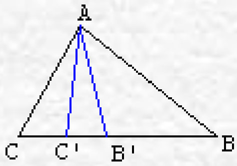
The side lengths of the hexagons are identical. The angles at P (right angle + angle between a & c) are identical. The angles at Q (right angle + angle between b & c) are identical. Therefore all four hexagons are identical.

Proof #16



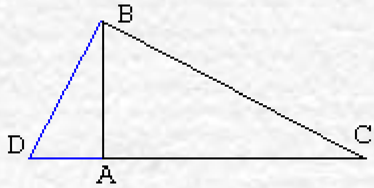
This proof appears in the Book IV of *Mathematical Collection* by Pappus of Alexandria (ca A.D. 300) [Eves, Pappas]. It generalizes the Pythagorean Theorem in two ways: the triangle ABC is not required to be right-angled and the shapes built on its sides are arbitrary parallelograms instead of squares. Thus build parallelograms CADE and CBFG on sides AC and, respectively, BC. Let DE and FG meet in H and draw AL and BM parallel and equal to HC. Then area(ABML)=area(CADE)+area(CBFG). Indeed, with the sheering transformation already used in proofs #1 and #12, area(CADE)=area(CAUH)=area(SLAR) and also area(CBFG)=area(CBVH)=area(SMBR). Now, just add up what's equal.

Proof #17



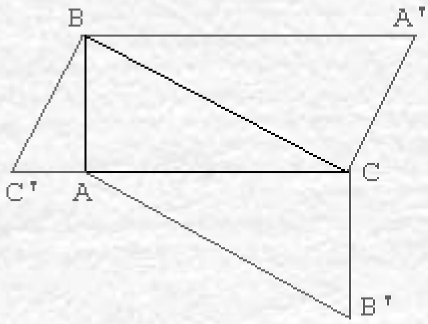
This is another generalization that does not require right angles. It's due to Tabit ibn Qorra (836-901). [Eves]. If angles CAB, AC'B and AB'C are equal then $AC^2 + AB^2 = BC(CB' + BC')$. Indeed, triangles ABC, AC'B and AB'C are similar. Thus we have AB/BC' = BC/AB and AC/CB' = BC/AC which immediately leads to the required identity. In case the angle A is right, the theorem reduces to the Pythagorean and the proof to the #6.

Proof #18



This proof is a variation on #6. On the small side AB add a right-angled triangle ABD similar to ABC. Then, naturally, DBC is similar to the other two. From area(ABD) + area(ABC) = area(DBC), $AD = AB^2/AC$ and $BD = AB \cdot BC/AC$ we derive $(ab^2/AC) \cdot AB + AB \cdot AC = (AB \cdot BC/AC) \cdot BC$. Dividing by AB/AC leads to $AB^2 + AC^2 = BC^2$.

Proof #19



This one is a cross between #7 and #19. Construct triangles ABC', BCA', and ACB' similar to ABC, as on the diagram. By construction, ABC = ACB'. In addition, triangles BCC' and BCA' are also equal. Thus we conclude that area(ACB') + area(ABC') = area(BCA'). From the similarity of triangles we get as before $AC' = AB^2/AC$ and $CA' = AB \cdot BC/AC$. Putting all together yields $(AB^2/AC) \cdot AB + AB \cdot AC = BC \cdot (AB \cdot BC/AC)$ which is the same as in #19.

Proof #20

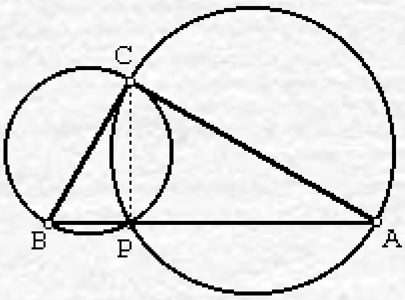
The following is an excerpt from a letter by Dr. Scott Brodie from the Mount Sinai School of Medicine, NY who sent me a couple of proofs of the theorem proper and its generalization to the Law of

Cosines:

> The first proof I merely pass on from the excellent discussion in the Project Mathematics series, based on Ptolemy's theorem on quadrilaterals inscribed in a circle: for such quadrilaterals, the sum of the products of the lengths of the opposite sides, taken in pairs equals the product of the lengths of the two diagonals. For the case of a rectangle, this reduces immediately to $a^2 + b^2 = c^2$.

Proof #21

Here is the second proof from Dr. Scott Brodie's letter.

> We take as known a "power of the point" theorems: If a point is taken exterior to a circle, and from the point a segment is drawn tangent to the circle and another segment (a secant) is drawn which cuts the circle in two distinct points, then the square of the length of the tangent is equal to the product of the distance along the secant from the external point to the nearer point of intersection with the circle and the distance along the secant to the farther point of intersection with the circle.
>
> 
>
>
> Let ABC be a right triangle, with the right angle at C. Draw the altitude from C to the hypotenuse; let P denote the foot of this altitude. Then since CPB is right, the point P lies on the circle with diameter BC; and since CPA is right, the point P lies on the circle with diameter AC. Therefore the intersection of the two circles on the legs BC, CA of the original right triangle coincides with P, and in particular, lies on AB. Denote by x and y the lengths of segments BP and PA, respectively, and, as usual let a, b, c denote the lengths of the sides of ABC opposite the angles A, B, C respectively. Then, $x + y = c$.
>
> Since angle C is right, BC is tangent to the circle with diameter CA, and the power theorem states that $a^2 = xc$; similarly, AC is tangent to the circle with diameter BC, and $b^2 = yc$. Adding, we find

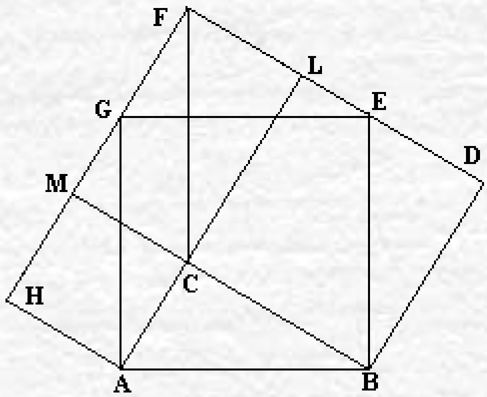
$a^2 + b^2 = xc + yc = c^2$, Q.E.D.

Dr. Brodie also created a Geometer's SketchPad file to illustrate this proof.

Proof #22

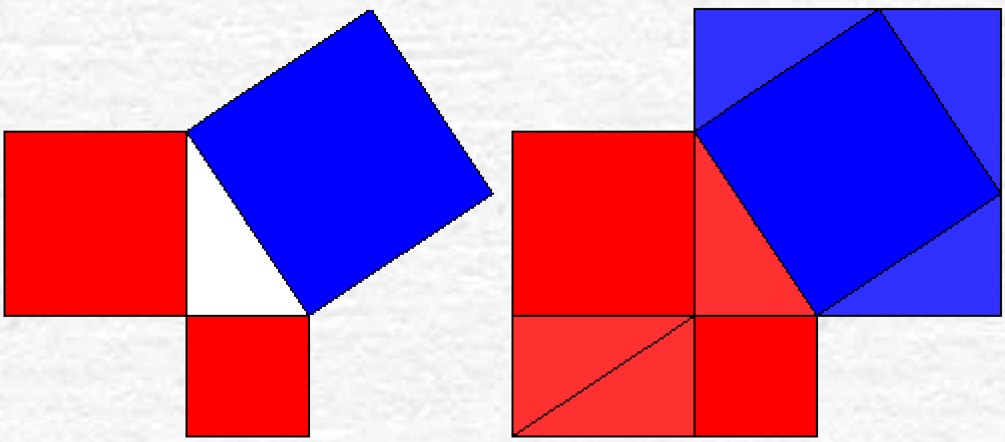
Another proof is based on the Heron's formula which I already used in Proof #7 to display triangle areas. This is a rather convoluted way to prove the Pythagorean Theorem that, nonetheless reflects on the centrality of the Theorem in the geometry of the plane.

Proof #23



[Swetz] ascribes this proof to abu' l'Hasan Thâbit ibn Qurra Marwân al'Harrani (826-901). It's the second of the proofs given by Thâbit ibn Qurra. The first one is essentially the #2 above.

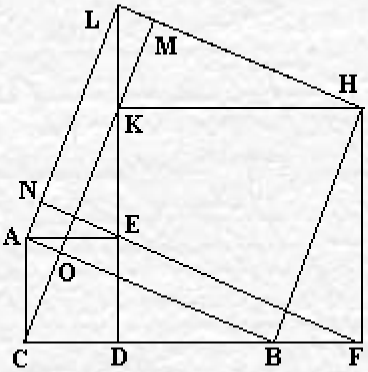
The proof resembles part 3 from proof #12. $\triangle ABC = \triangle FLC = \triangle FMC = \triangle BED = \triangle AGH = \triangle FGE$. On one hand, the area of the shape ABDFH equals $AC^2 + BC^2 + \text{area}(\triangle ABC + \triangle FMC + \triangle FLC)$. On the other hand, $\text{area}(ABDFH) = AB^2 + \text{area}(\triangle BED + \triangle FGE + \triangle AGH)$.



This is an "unfolded" variant of the above proof. Two pentagonal regions - the red and the blue - are obviously equal and leave the same area upon removal of three equal triangles from each.

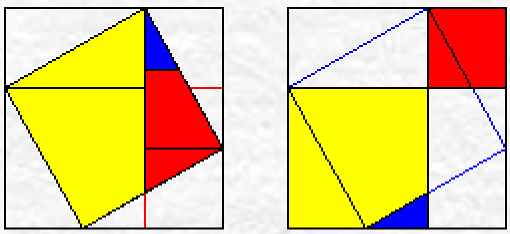
The proof is popularized by Monty Phister, author of the inimitable *Gnarly Math* CD-ROM.

Proof #24



B.F.Yanney (1903, [Swetz]) gave a proof using the "sliding argument" also employed in the Proofs #1 and #12. Successively, areas of LMOA, LKCA, and ACDE (which is AC^2) are equal as are the areas of HMOB, HKCB, and HKDF (which BC^2). BC = DF. Thus $AC^2 + BC^2$ = area(LMOA) + area(HMOB) = area(ABHL) = AB^2.

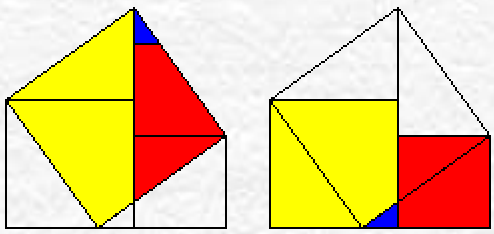
Proof #25



This proof I discovered at the site maintained by Bill Casselman where it is presented by a Java applet. (The site has since disappeared.)

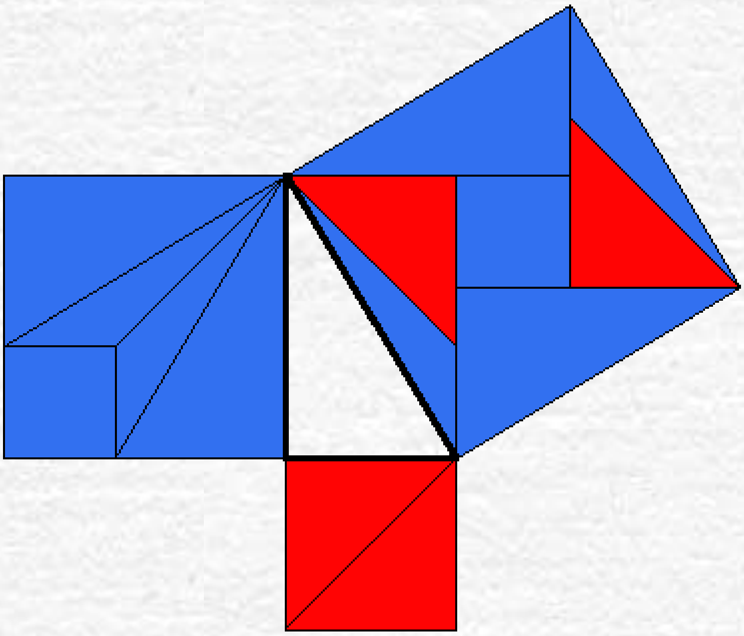
With all the above proofs, this one must be simple. Similar triangles like in proofs #6 or #13.

Proof #26



The same pieces as in proof #26 may be rearrangened in yet another manner.

Proof #27



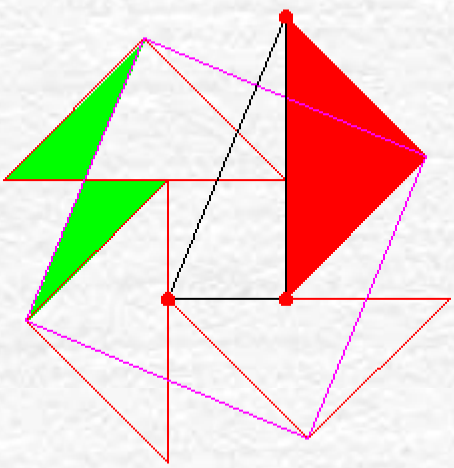
Melissa Running from MathForum has kindly sent me a link to A proof of the Pythagorean Theorem by Liu Hui (third century AD). The page is maintained by Donald B. Wagner, an expert on history of science and technology in China. The diagram is a reconstruction from a written description of an algorithm by Liu Hui (third century AD). For details you are referred to the original page.

Proof #28

A mechanical proof of the theorem deserves a page of its own.

Pertinent to that proof is a page "Extra-geometric" proofs of the Pythagorean Theorem by Scott Brodie

Proof #29

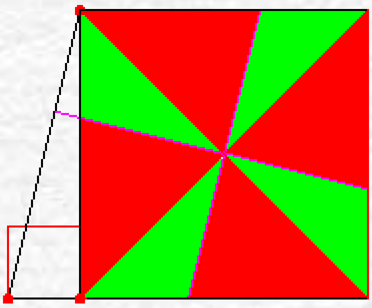


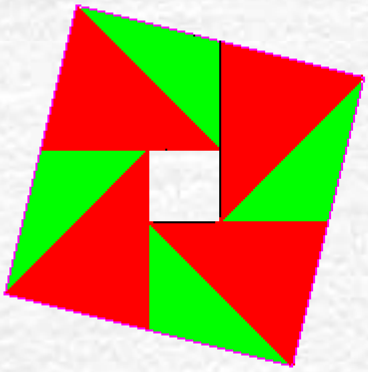
This proof I found in R. Nelsen's sequel *Proofs Without Words II*. (It's due to Poo-sung Park and was originally published in *Mathematics Magazine, Dec 1999*). Starting with one of the sides of a right triangle, construct 4 congruent right isosceles triangles with hypotenuses of any subsequent two perpendicular and apices away from the given triangle. The hypotenuse of the first of these triangles (in red in the diagram) should coincide with one of the sides.

The apices of the isosceles triangles form a square with the side equal to the hypotenuse of the given triangle. The hypotenuses of those triangles cut the sides of the square at their midpoints. So that there appear to be 4 pairs of equal triangles (one of the pairs is in green). One of the triangles in the pair is inside the square, the other is outside. Let the sides of the original triangle be a, b, c (hypotenuse). If the first isosceles triangle was built on side b, then each has area $b^2/4$. We obtain

$$a^2 + 4b^2/4 = c^2$$

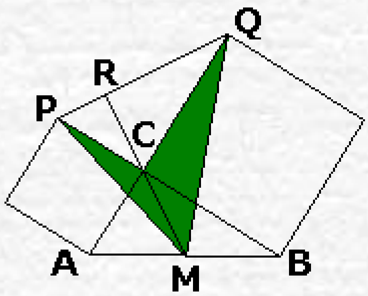
Here's a dynamic illustration and another diagram that shows how to dissect two smaller squares and rearrange them into the big one.





Proof #30

Given right △ABC, let, as usual, denote the lengths of sides BC, AC and that of the hypotenuse as a, b, and c, respectively. Erect squares on sides BC and AC as on the diagram. According to SAS, triangles ABC and PCQ are equal, so that ∡QPC = ∡A. Let M be the midpoint of the hypotenuse. Denote the intersection of MC and PQ as R. Let's show that MR ⊥ PQ.



The median to the hypotenuse equals half of the latter. Therefore, △CMB is isosceles and ∡MBC = ∡MCB. But we also have ∡PCR = ∡MCB. From here and ∡QPC = ∡A it follows that angle CRP is right, or MR $\perp$ PQ.

With these preliminaries we turn to triangles MCP and MCQ. We evaluate their areas in
two different ways:One one hand, the altitude from M to PC equals AC/2 = b/2.
But also PC = b. Therefore, Area($\triangle$MCP) = $b^2/4$.
On the other hand, Area($\triangle$MCP) = CM·PR/2 = c·PR/4. Similarly, Area($\triangle$MCQ) = $a^2/4$ and
also Area($\triangle$MCQ) = CM·RQ/2 = c·RQ/4.

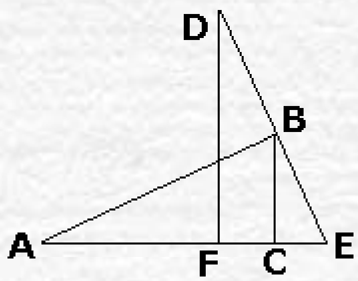
We may sum up the two identities: $a^2/4 + b^2/4 = c\cdot PR/4 + c\cdot RQ/4$, or $a^2/4 + b^2/4 = c\cdot c/4$.

(My gratitude goes to Floor van Lamoen who brought this proof to my attention.
It appeared in *Pythagoras* - a dutch math magazine for schoolkids - in the December 1998 issue, in an article by Bruno Ernst. The proof is attributed
to an American High School student from 1938 by the name of Ann Condit.)

Proof #31

Let ABC and DEF be two congruent right triangles such that B lies on DE and A, F, C, E are
collinear. BC = EF = a, AC = DF = b, AB= DE = c. Obviously, AB $\perp$ DE. Compute the area
of $\triangle$ADE in two different ways.

Area($\triangle$ADE) = AB·DE/2 = $c^2/2$ and
also Area($\triangle$ADE) = DF·AE/2 = b·AE/2. AE = AC + CE = b + CE. CE
can be found from similar triangles BCE and DFE: CE = BC·FE/DF = a·a/b.
Putting things together
we obtain



$$c^2/2 = b(b + a^2/b)/2$$

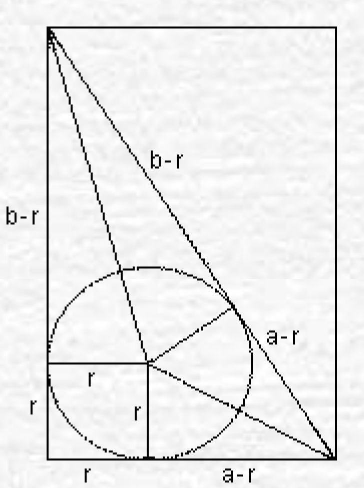
(This proof is a simplification of one of the proofs by Michelle Watkins, a student at the

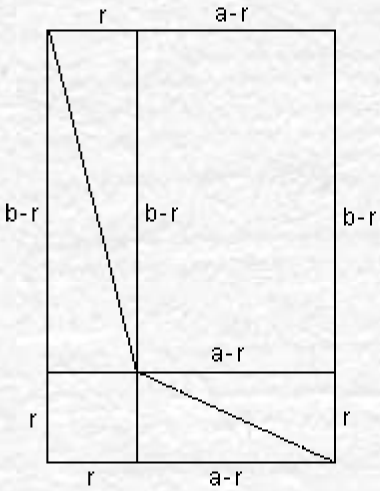
University of North Florida, that appeared in *Math Spectrum* 1997/98, v30, n3, 53-54.)

The next two proofs have accompanied the following message from Shai Simonson,
Professor at Stonehill College in Cambridge, MA:

> Greetings,
>
> I was enjoying looking through your site, and stumbled on the long list of Pyth Theorem Proofs.
>
> In my course "The History of Mathematical Ingenuity" I use two proofs that use an inscribed circle in a right triangle. Each proof uses two diagrams, and each is a different geometric view of a single algebraic proof that I discovered many years ago and published in a letter to Mathematics Teacher.
>
> The two geometric proofs require no words, but do require a little thought.
>
> Best wishes,
>
> Shai

Proof #32

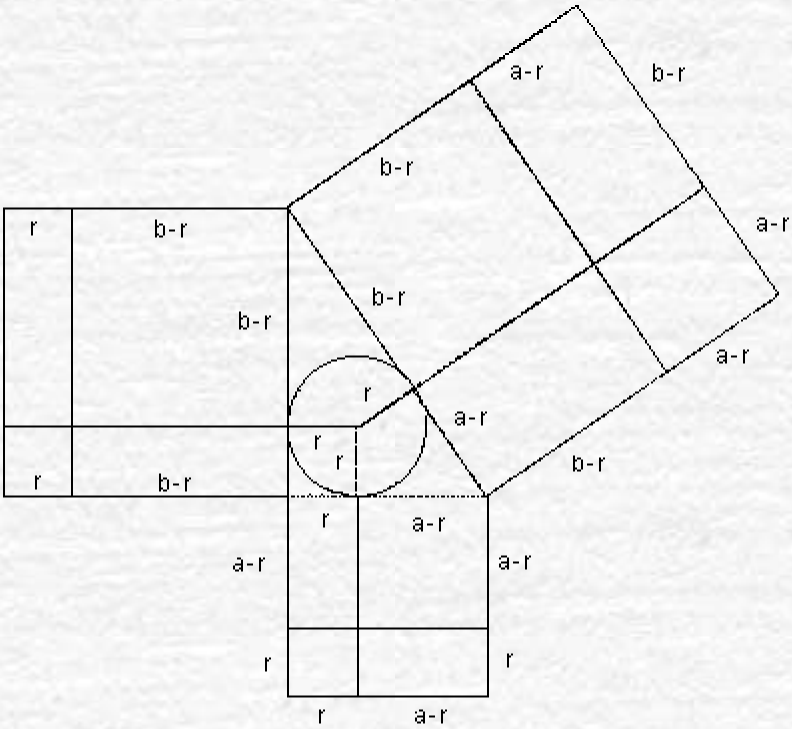




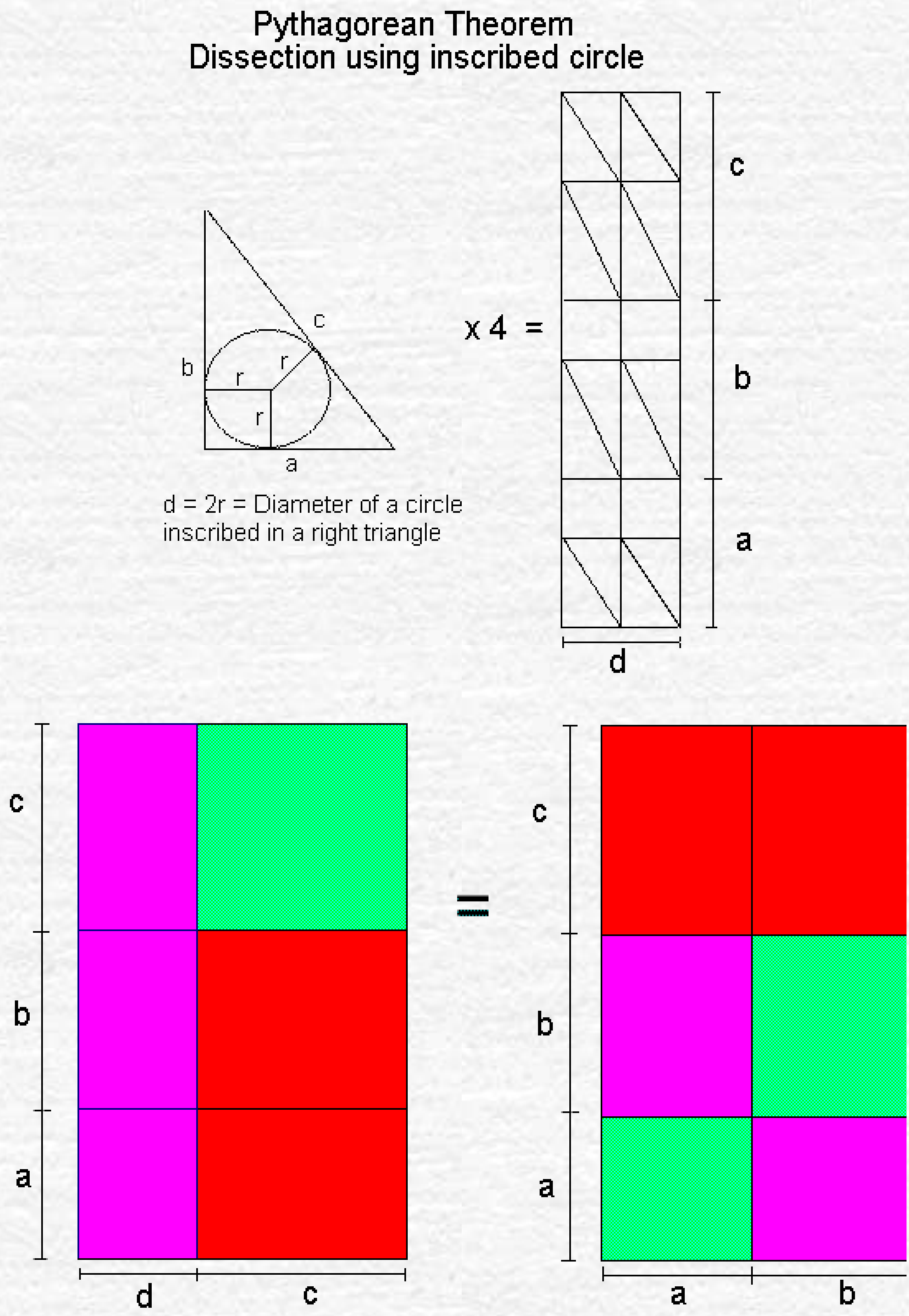
$$r^2 + r(a - r) + r(b - r) = ab/2$$

$$r(a + b - r) = ab/2$$

$$(a - r)(b - r) = ab/2$$

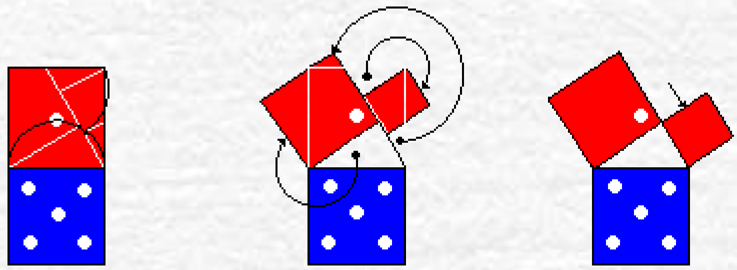


Proof #33



Proof #34

Cracked Domino - a proof by Mario Pacek (aka Pakoslaw Gwizdalski) -
also requires some thought.



A B C

The proof sent via email was accompanied by the following message:

> This new, extraordinary and extremely elegant proof of quite probably the most fundamental theorem in mathematics (hands down winner with respect to the # of proofs 367?) is superior to all known to science including the Chinese and James A. Garfield's (20th US president), because it is direct, does not involve any formulas and even preschoolers can get it. Quite probably it is identical to the lost original one - but who can prove that? Not in the Guinness Book of Records yet!

The manner in which the pieces are combined may well be original. The dissection itself is
well known (see Proofs 26 and 27) and is described in Frederickson's book, p. 29.
It's remarked there that B. Brodie (1884) observed that the dissection like that also
applies to similar rectangles. The dissection is also a particular instance of
the superposition proof by K.O.Friedrichs.

Proof #35

This proof is due to J. E. Böttcher and has been quoted by Nelsen (*Proofs Without Words II*, p. 6).

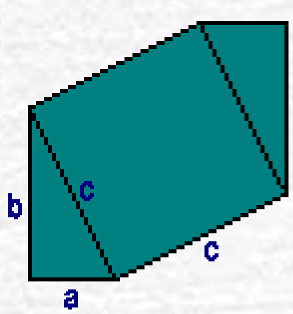
I think cracking this proof without words is a good exercise for middle or high
school geometry class.

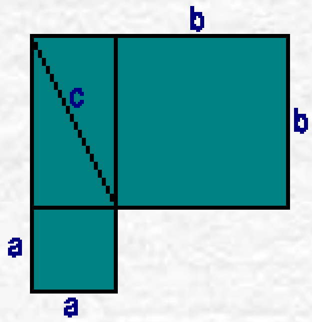
Proof #36

An applet by David King that demonstrates this proof has been placed on a
separate page.

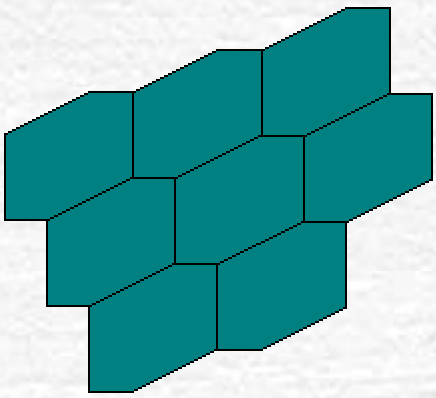
Proof #37

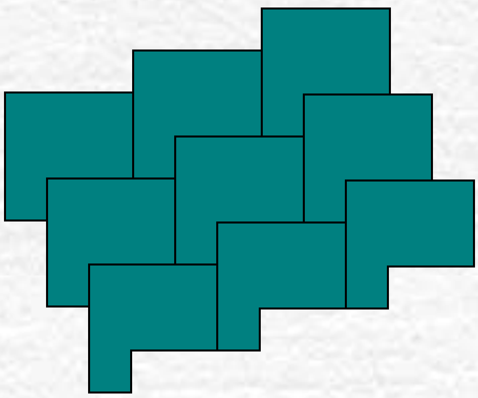
This proof was also communicated to me by David King. Squares and 2 triangles combine to
produce two hexagon of equal area, which might have been established as
in Proof #9.However, both hexagons tessellate the plane.





Both hexagons tessellate:

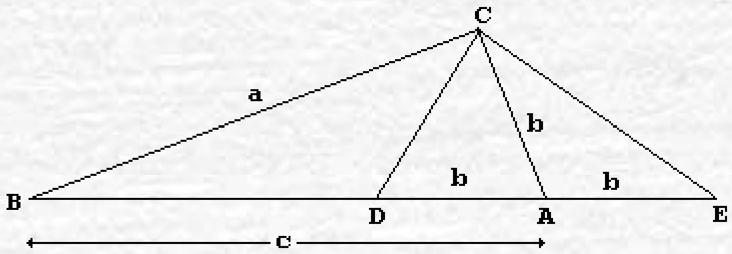




For every hexagon in the left tessellation there is a hexagon in the right tessellation.
Both tessellations have the same lattice structure which is demonstrated by an applet.
The Pythagorean theorem is proven after two triangles are removed from
each of the hexagons.

Proof #38

(By J. Barry Sutton, The Math Gazette, v 86, n 505, March 2002, p72.)



Let in △ABC, angle C = 90^{o}. As usual, AB = c, AC = b, BC = c. Define points D and E on AB so that AD = AE = b.

By construction, C lies on the circle with center A and radius b. Angle DCE subtends its diameter and thus is right: ∡DCE = 90^{o}. It follows that ∡BCD = ∡ACE. Since △ACE is isosceles, ∡CEA = ∡ACE.

Triangles DBC and EBC share ∡DBC. In addition, ∡BCD = ∡BEC. Therefore, triangles DBC and EBC are similar. We have BC/BE = BD/BC, or

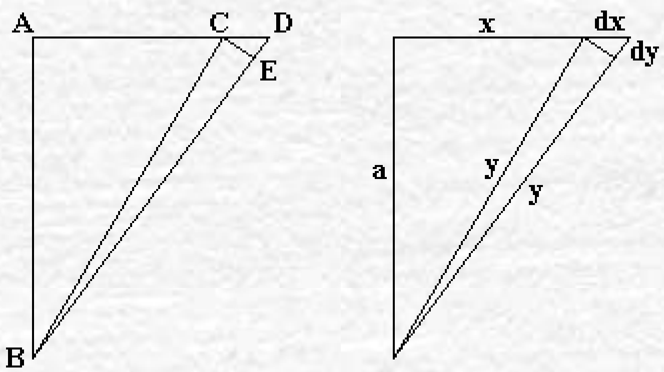
$$a / (c + b) = (c - b) / a.$$

An finally

$$a^2 = c^2 - b^2,$$
$$a^2 + b^2 = c^2.$$

The diagram reminds one of Tabit ibn Qorra's proof. But the two are quite different.

Proof #39



This one is by Michael Hardy from University of Toledo and was published in
The Mathematical Intelligencer in 1988. It must be taken with a grain of salt.

Let ABC be a right triangle with hypotenuse BC. Denote AC = x and BC = y.
Then, as C moves along the line AC, x changes and so does y. Assume x changed by a
small amount dx. Then y changed by a small amount dy. The triangle CDE may
be approximately considered right. Assuming it is, it shares one angle (D)
with triangle ABD, and is therefore similar to the latter. This leads to
the proportion x/y = dy/dx, or a (separable) differential equation

$$y\cdot dy - x\cdot dx = 0,$$

which after integration gives $y^2 - x^2 = \text{const}$. The value of the constant is determined from
the initial condition for x = 0. Since y(0) = a, $y^2 = x^2 + a^2$ for all x.

It is easy to make an issue with this proof. What does it mean for a triangle to be
approximately right? I can offer the following explanation. Triangles ABC and ABD
are right by construction. We have, $AB^2 + AC^2 = BC^2$ and also $AB^2 + AD^2 = BD^2$, by
the Pythagorean theorem. In terms of x and y, the theorem appears as

$$x^2 + a^2 = y^2$$
$$(x + dx)^2 + a^2 = (y + dy)^2$$

which, after subtraction, gives

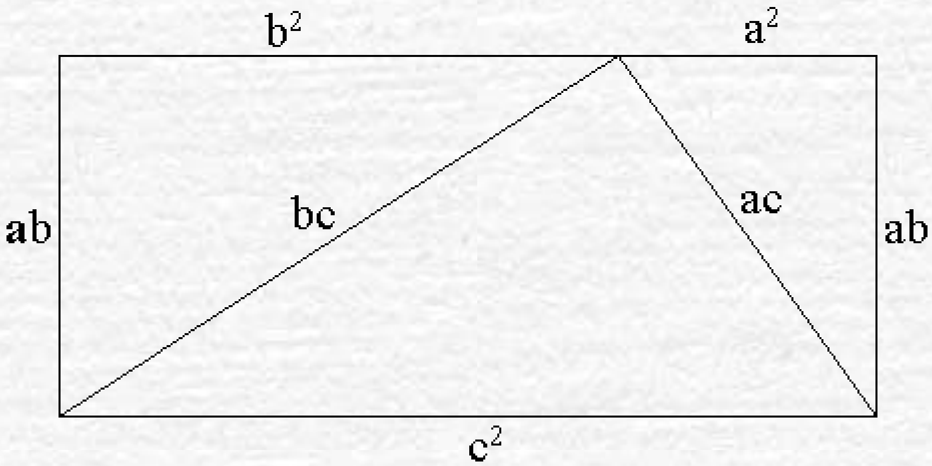
$$y\cdot dy - x\cdot dx = (dx^2 - dy^2)/2.$$

For small dx and dy, dx^2 and dy^2 are even smaller and might be neglected,
leading to the approximate $y\cdot dy - x\cdot dx = 0$.■

The trick in Michael's vignette is in skipping the issue of approximation.
But can one really justify the derivation without relying on the Pythagorean theorem in the first place? Regardless,

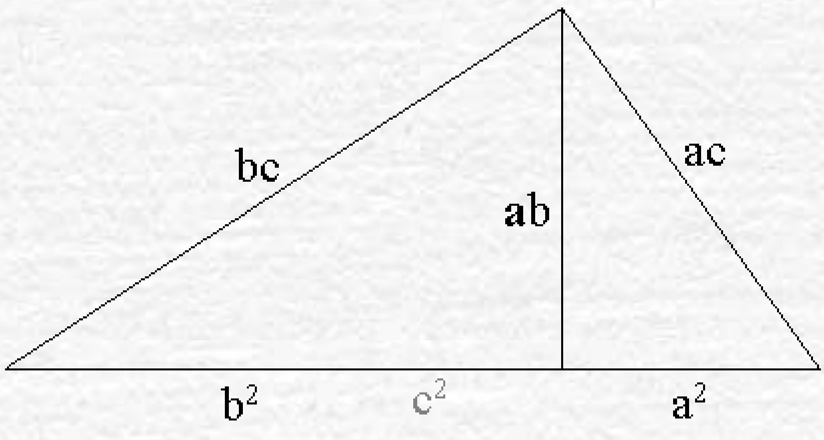
I find it very much to my enjoyment to have the ubiquitous equation $y \cdot dy - x \cdot dx = 0$ placed in
that geometric context.

Proof #40



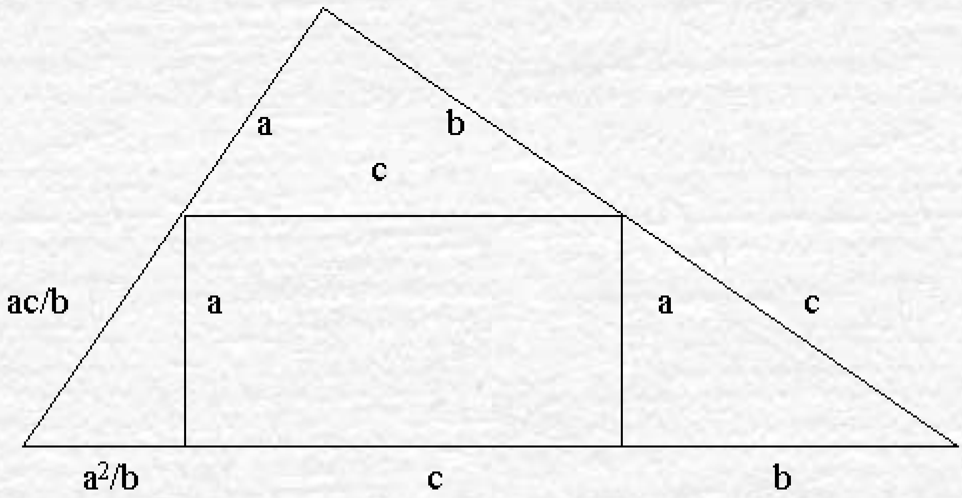
This one was sent to me by Geoffrey Margrave from Lucent Technologies.
It looks very much as #8, but is arrived at in a different way. Create 3 scaled copies
of the triangle with sides a, b, c by multiplying it by a, b, and c in turn.
Put together, the three similar triangles thus obtained form a rectangle whose
upper side is $a^2 + b^2$, whereas the lower side is c^2.
(Which also shows that #8 might have been concluded in a shorter way.)

Also, picking just two triangles leads to a variant of Proofs #6 and #19:



In this form the proof appears in [Birkhoff, p. 92].

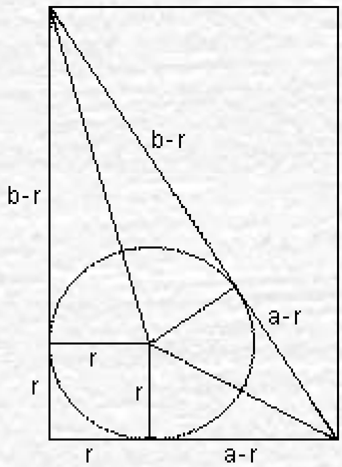
Yet another variant that could be related to #8 has been sent by James F.:



The latter has a twin with a and b swapping their roles.

Proof #41

The proof is based on the same diagram as #33 [Pritchard, p. 226-227].



Area of a triangle is obviously rs, where r is the incircle and s = (a + b + c)/2 the
semiperimeter of the triangle. From the diagram, the hypothenuse c = (a - r) + (b - r),
or r = s - c. The area of the triangle then is computed in two ways:

$$s(s - c) = ab/2,$$

which is equivalent to

$$(a + b + c)(a + b - c) = 2ab,$$

or

$$(a + b)^2 - c^2 = 2ab.$$

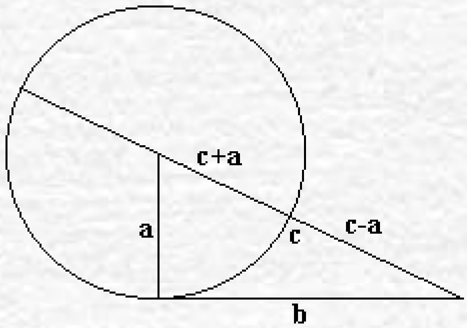
And finally

$$a^2 + b^2 - c^2 = 0.$$

(The proof is due to Jack Oliver, and was originally published in *Mathematical Gazette* **81**
(March 1997), p 117-118.)

Απόδειξη #42

[Pritchard, p. 229].



Εφαρμόστε τη δύναμη σημείου στο διάγραμμα ανωτέρω όπου η πλευρά b
είναι εφαπτομένη σε έναν κύκλο ακτίνας a:
$(c - a)(c + a) = b^2$.
Το αποτέλεσμα ακολουθεί αμέσως. .

Αναφορές

1. G. D. Birkhoff and R. Beatley, *Basic Geometry*, AMS Chelsea Pub, 2000

2. W. Dunham, *The Mathematical Universe*, John Wiley & Sons, NY, 1994.
3. W. Dunham, *Journey through Genius*, Penguin Books, 1991
4. H. Eves, *Great Moments in Mathematics Before 1650*, MAA, 1983
5. G. N. Frederickson, *Dissections: Plane & Fancy*, Cambridge University Press, 1997
6. R. B. Nelsen, *Proofs Without Words*, MAA, 1993
7. R. B. Nelsen, *Proofs Without Words II*, MAA, 2000
8. J. A. Paulos, *Beyond Numeracy*, Vintage Books, 1992
9. T. Pappas, *The Joy of Mathematics*, Wide World Publishing, 1989
10. C. Pritchard, *The Changing Shape of Geomtetry*, Cambridge University Press, 2003
11. F. J. Swetz, *From Five Fingers to Infinity*, Open Court, 1996, third printing

On Internet

1. Pythagoras, biography
2. Ask Dr. Math
   - Another incarnation of #4
   - They try and try and try...
   - President Garfield's
3. Eric's Treasure Trove features more than 10 proofs
4. A proof of the Pythagorean Theorem by Liu Hui (third century AD)
   An interesting page from which I borrowed Proof #28
5. An animated reincarnation of #9